

# Improved Guarantees for Vertex Sparsification in Planar Graphs

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## Abstract

Graphs Sparsification aims at compressing large graphs into smaller ones while (approximately) preserving important characteristics of the input graph. In this work we study Vertex Sparsifiers, i.e., sparsifiers whose goal is to reduce the number of vertices. More concretely, given a weighted graph  $G = (V, E)$ , and a terminal set  $K$  with  $|K| = k$ , a quality  $q$  vertex cut sparsifier of  $G$  is a graph  $H$  such that  $K \subseteq V(H)$  and for any bipartition  $U, K \setminus U$  of the terminal set  $K$ , the values of minimum cut separating  $U$  from  $K \setminus U$  in  $G$  and  $H$  are within a factor  $q$  from each other. Similarly, we define vertex flow and distance sparsifiers that (approximately) preserve multicommodity flows and distances among terminal pairs, respectively.

We study such vertex sparsifiers for planar graphs. We show that if all the  $k$  terminals in  $K$  lie on the same face of the input planar graph  $G$ , then there exist quality 1 vertex cut, flow and distance sparsifiers of size  $O(k^2)$  that are also planar. This improves upon the previous best known bound  $O(k^2 2^{2k})$  for cut and flow sparsifiers for such class of graphs by an exponential factor. Our upper bound for cut sparsifiers also matches the known lower bound  $\Omega(k^2)$  on the number of edges of such sparsifiers for this class of  $k$ -terminal planar graphs. We also provide a new lower bound of  $\Omega(k^{1+1/(t-1)})$  on the size of any data structure that approximately preserves the pairwise terminal distance in sparse *general* graphs within a multiplicative factor of  $t$  or an additive error  $2t - 3$ , for any  $t \geq 2$ .

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# 1 Introduction

Very large graphs or networks are ubiquitous nowadays, from social networks to information networks. One natural and effective way of processing and analyzing such graphs is to compress or sparsify the graph into a smaller one that well preserves certain properties of the original graph. Such a sparsification can be obtained by reducing the number of *edges*. Typical examples include cut sparsifiers [5], spectral sparsifiers [34] and spanners [36], which are subgraphs defined on the same vertex set of the original graph  $G$  while having much smaller number of edges and still well approximating the cut structure, spectral properties and pairwise distance of  $G$ , respectively. Another way of performing sparsification is by reducing the number of *vertices*, which is most appealing when only the properties among a subset of vertices (which are called *terminals*) are of interest (see e.g., [32, 3, 23]). We call such small graphs *vertex sparsifiers* of the original graph, which will be the main focus of this paper.

More specifically, given a capacitated undirected graph  $G = (V, E, c)$ , and a set of terminals  $K$ , we are looking for a graph  $H = (V_H, E_H, c_H)$  with as few number of vertices as possible and  $K \subseteq V_H$  such that the properties (e.g., connectivity) or quantities (e.g., cut value, distance) among vertices in  $K$  in  $H$  are the same as or close to the corresponding properties or quantities in  $G$ . If  $|K| = k$ , we will also call the graph  $G$  a *k-terminal graph*. Depending on the quantities that  $H$  is preserving, we will consider the following three types of vertex sparsifiers.

- If for every bipartition  $U, K \setminus U$  of the terminal set  $K$ , the value of minimum cut separating  $U$  from  $K \setminus U$  in  $G$  is within a factor of  $q$  of the value of minimum cut separating  $U$  from  $K \setminus U$  in  $H$ , then  $H$  is called a *quality  $q$  (vertex) cut sparsifier* of  $G$ . If  $H$  is a quality 1 cut sparsifier, then it will be also called a *mimicking network* [20].
- If all the multicommodity flows that can be routed between the terminals in  $H$  are preserved as in  $G$ , up to a factor of  $q \geq 1$ , then  $H$  is called a *quality  $q$  (vertex) flow sparsifier* of  $G$ .
- If for every two terminals  $s, t \in K$ , the shortest-path distance between  $s, t$  in  $H$  is within a factor of  $q$  of the shortest-path distance between  $s, t$  in  $G$ , then the graph  $H$  is called a *quality  $q$  (vertex) distance sparsifier* of  $G$ .

There has been a long line of work on investigating the tradeoff between the quality of the vertex sparsifier and its size (see e.g., [15, 24, 3] and Section 1.2). (Throughout, cut, flow and distance sparsifiers will refer to their vertex versions.) We are particularly interested in quality 1 vertex sparsifiers, which are sparsifiers that exactly preserves the corresponding quantities. Quality 1 *cut sparsifiers* (or equivalently, mimicking networks) were first introduced by Hagerup et al. [20], who proved that for any graph  $G$ , there always exists a mimicking network of size  $O(2^{2^k})$ , where  $k$  is the size of terminal set  $K$ . Chaudhuri et al. [8] then constructed mimicking networks with  $O(k)$  vertices for bounded treewidth graphs and outerplanar graphs, and they also give a lower bound of  $k + 1$  for the size of such networks, that even holds for a star graph [8]. Krauthgamer and Rika [24] showed how to build a mimicking network of size  $O(k^2 2^{2k})$  for any planar graph  $G$  by employing the structure of subgraphs of cycles of the dual graph of  $G$  and Euler's formula. They also proved a lower bound of  $\Omega(k^2)$  on the number of edges of the mimicking network of planar graphs, and a lower bound of  $2^{\Omega(k)}$  on the number of vertices of the mimicking network for general graphs. Khan and Raghavendra [22] gave an improved upper bound (while still doubly exponential in  $k$ ) for general graphs as well as a lower bound of  $2^{(k-1)/2}$  for general graphs. They also give improved upper bounds (while still  $O(k)$ ) for trees and graphs with bounded treewidth.

Quality 1 vertex flow sparsifiers have been studied by Andoni, Gupta and Krauthgamer [3], who showed the existence of such sparsifiers of size  $O(k)$  for outerplanar and series-parallel graphs, and

$O(k^2 2^{2k})$  for planar graphs where all terminals lie on the same face. Goranci and Räcke [18] showed that unweighted quasi-bipartite graphs admit quality 1 vertex flow sparsifiers of size  $O(2^k)$ . It is not known if any general undirected graph  $G$  admits a constant quality *flow sparsifier* with size independent of  $|V(G)|$  and the edge capacities. For the quality 1 *distance sparsifier*, note that if there is no restriction on the structure of the sparsifier, then such sparsifier of size  $k$  exists for every graph  $G$  (which will be the complete graph on terminals with appropriately chosen edge weights). Krauthgamer, Nguyen and Zondiner [23] showed that if this sparsifier is required to be a minor of the original graph, then one can construct such a distance-preserving minor of size  $O(k^4)$  for general graphs and there is a lower bound of  $\Omega(k^2)$  of the size of such a minor for planar graphs. The main reason of requiring the distance sparsifier  $H$  being a “minor” of the original graph  $G$  is to make sure that  $H$  is structurally similar to  $G$ . Here we relax the requirement for planar graphs  $G$  by only restricting the distance sparsifier  $H$  to be planar (not necessarily a minor of  $G$ ; see [19] for similar requirement for trees).

## 1.1 Our Results

We provide new constructions for quality 1 (exact) cut, flow and distance sparsifiers for  $k$ -terminal planar graphs, where all the terminals are assumed to lie on the same face. We call such  $k$ -terminal planar graphs *Okamura-Seymour (OS) instances*. They are of particular interest in the algorithm design and optimization community, due to the classical Okamura-Seymour theorem that characterizes the existence of feasible concurrent flows in such graphs (see e.g., [33, 10, 11, 28]).

We show that the size of quality 1 sparsifiers can be as small as  $O(k^2)$  for such instances, for which only exponential (in  $k$ ) size of cut and flow sparsifiers were known before [24, 3]. Formally, we have the following theorem.

**Theorem 1.1.** *For any OS instance  $G$ , i.e., a  $k$ -terminal planar graph in which all terminals lie on the same face, there exist quality 1 vertex cut, flow and distance sparsifiers of size  $O(k^2)$ . Furthermore, the resulting sparsifiers are also planar.*

We remark that all the above sparsifiers can be constructed in polynomial time (in  $n$  and  $k$ ), but we will not optimize the running time here. As we mentioned above, previously the only known upper bound on the size of quality 1 cut and flow sparsifiers for OS instance was  $O(k^2 2^{2k})$ , given by [24, 3]. Our upper bound for cut sparsifier also matches the lower bound of  $\Omega(k^2)$  for OS instance given by [24]. More specifically, in [24], an OS instance (that is a grid in which all terminals lie on the boundary) is constructed, and used to show that any mimicking network for this instance needs  $\Omega(k^2)$  edges, which is thus a lower bound for planar graphs. Finally, we remark that even though our distance sparsifier is not necessarily a minor of the original graph  $G$ , it still shares the nice property of being planar as  $G$ . It is worth mentioning that in [25], it is proven that there exists a  $k$ -terminal planar graph  $G$  (not necessarily an OS instance), such that any quality 1 distance sparsifier of  $G$  that is planar requires at least  $\Omega(k^2)$  vertices.

Our second result is a lower bound on the size of any *data structure* (not necessarily a graph) that approximately preserves pairwise terminal distances of *general*  $k$ -terminal graphs, which provides a trade-off between the distance stretch and the space complexity.

**Theorem 1.2.** *For every  $\varepsilon > 0$  and  $t \geq 2$ , there exists a (sparse)  $k$ -terminal  $n$ -vertex graph such that  $k = o(n)$ , and any compression function that approximates pairwise terminal distances within a multiplicative factor of  $t - \varepsilon$  or an additive error  $2t - 3$  must use  $\Omega(k^{1+1/(t-1)})$  bits.*

Note that for any  $k$ -terminal graph  $G$ , as we mentioned above, if we do not have any restriction on the structure of the distance sparsifier, then  $G$  always admits a trivial quality 1 distance sparsifier

Type of sparsifier	Graph family	Upper Bound	Lower Bound
Cut	General	$O(2^{2^k})$ [20]	$2^{\Omega(k)}$ [24, 22]
Cut	Planar	$O(k^2 2^{2k})$ [24]	
Cut	Planar OS	$O(k^2)$ <b>[new]</b>	$ E(G')  \geq \Omega(k^2)$ [24]
Flow	Outerplanar	$O(k)$ [8]	$k + 1$ [8]
Flow	Planar OS	$O(k^2 2^{2k})$ [3]	follows from cut
Flow	Planar OS	$O(k^2)$ <b>[new]</b>	follows from cut
Distance (minor)	General	$O(k^4)$ [23]	
Distance (minor)	Planar OS		$\Omega(k^2)$ [23]
Distance (planar)	Planar OS	$O(k^2)$ <b>[new]</b>	

Table 1: Overview on the current best trade-offs for quality 1 vertex sparsifiers.

$H$  which is the complete weighted graph on  $k$  terminals with each edge weight being equal to the distance between the two endpoints in  $G$ . Furthermore, by the well-known result of Awerbuch [4], such a graph  $H$  in turn admits a multiplicative  $(2t - 1)$ -*spanner*  $H'$  with  $O(k^{1+1/t})$  edges, that is, all the distances in  $H$  are preserved up to a multiplicative factor of  $2t - 1$  in  $H'$ , for any  $t \geq 1$ . This directly implies that the  $k$ -terminal graph  $G$  has a quality  $2t - 1$  distance sparsifier with  $k$  vertices and  $O(k^{1+1/t})$  edges. On the other hand, though *unconditional* lower bounds of type similar to Theorem 1.2 have been known for the number of edges of spanners [26, 27, 39], we are not aware of such lower bounds for the size of *data structure* that preserves pairwise terminal distances for any  $k$ -terminal  $n$ -vertex graph when  $k = o(n)$ . In the extreme case when  $k = n$  (i.e., all the vertices are terminals), the recent work by Abboud and Bodwin [1] shows that any data structure that preserves the distances with an additive error  $t$  needs  $\Omega(n^{4/3-\varepsilon})$  bits, for any  $\varepsilon > 0, t = O(n^\delta)$  and  $\delta = \delta(\varepsilon)$  (see also the follow-up work [2]).

## 1.2 Our Techniques

We construct our quality 1 cut and distance sparsifiers by repeatedly performing *Wye-Delta transformations*, which are local operations that preserve cut values and distances and have proven very powerful in analyzing electrical networks and in the theory of circular planar graphs (see e.g., [14, 16]). Kahn and Raghavendra [22] used Wye-Delta transformations to construct quality 1 cut sparsifiers of size  $O(k)$  for trees, while our case (i.e., the planar OS instances) is more general and complicated and previously it was not clear at all how to apply such transformations to a more broad class of graphs.

Our approach is as follows. Given a  $k$ -terminal planar graph with terminals lying on the outer face, we first embed it into some large grid with terminals lying on the boundary of the grid. Subsequently, we show how to embed the resulting grid into a “more suitable” graph, which we will refer to as “half-grid”. Finally, using the Wye-Delta operations, we reduce the “half-grid” into another graph whose number of vertices can be bounded by  $O(k^2)$ . Since we shall argue that the above graph reductions preserve exactly all terminal minimum cuts, our result follows.

The distance sparsifiers can be constructed similarly by slightly modifying the Wye-Delta operation. Our flow sparsifiers follow from the construction of cut sparsifiers and the flow/cut gaps

for OS instances (which has also been observed by Andoni et al. [3]).

Our lower bound of the space complexity of any compression function approximately preserving terminal pairwise distance is derived by combining extremal combinatorics construction of Steiner Trippe System that was used to prove lower bounds on the size of distance approximating minors (see [12]) and the incompressibility technique from [31].

### 1.3 Other Related Work

There are many other known tradeoffs between the sparsifier's quality  $q$  and its size for  $q > 1$ , i.e., when the sparsifiers only *approximately* preserve the corresponding properties. Chuzhoy [13] gives a constant quality cut and flow sparsifiers with size depending on the total capacity of the edges on the terminals. Andoni et al. [3] constructed quality  $(1 + \varepsilon)$  flow sparsifier of size  $\text{poly}(k/\varepsilon)$  for quasi-bipartite networks and quality  $O(\log t / \log \log t)$  flow sparsifier of size  $O(t \cdot \text{poly}(k))$  for  $G$  with bounded treewidth  $t$ . (They also constructed a *sketch* (not a graph) of quality  $1 + \varepsilon$  and size  $f(k, \varepsilon)$  (independent of  $n$ ) for general graphs.)

For sparsifiers with exactly  $k$  vertices (that is, supported only on the terminals), there are efficient algorithms for constructing quality  $O(\log k / \log \log k)$  cut and flow sparsifiers ([32, 29, 7, 15, 30]), and there exists a lower bound of  $\tilde{\Omega}(\sqrt{\log k})$  on the quality of such cut or flow sparsifiers [30] (see also [7, 29, 15]).

For vertex distance sparsifiers, there has been work on constructing *minors* that approximately preserve the pairwise distances among terminals for both general graphs and planar graphs (see [19, 9, 6, 15, 21, 12]).

## 2 Preliminaries

Let  $G = (V, E, c)$  be an undirected graph with terminal set  $K \subset V$  of cardinality  $k$ , where  $c : E \rightarrow \mathbb{R}_{\geq 0}$  assigns a non-negative capacity to each edge. We will refer to such a graph as a *k-terminal graph*. Throughout the paper we will be dealing with two special types of graphs.

A *grid* graph is a graph with  $n \times n$  vertices  $\{(u, v) : u, v = 1, \dots, n\}$ , where  $(u, v)$  and  $(u', v')$  are adjacent if  $|u' - u| + |v' - v| = 1$ . For  $k < n$ , a *half-grid* graph with  $k$  terminals is a graph  $T_k^n = (V, E)$  with  $K \subset V$  and  $n(n+1)/2$  vertices  $\{(i, j) : i \leq j \text{ and } i, j = 1, \dots, n\}$ , where  $(i, j)$  and  $(i', j')$  are connected by an edge if  $|i' - i| + |j' - j| = 1$ , and additional diagonal edges between  $(i, i)$  and  $(i+1, i+1)$  for  $i = 1, \dots, n-1$ . Moreover, each terminal vertex in  $T_k^n$  must be one of its diagonal vertices, i.e., every  $x \in K$  is of the form  $(m, m)$  for some  $m \in \{1, \dots, n\}$ . Let  $\hat{T}_k^n$  be the same graph as  $T_k^n$  but excluding the diagonal edges.

We present three different ways to sparsify the number of vertices in  $G$  depending on the feature we want to preserve.

Let  $U \subset V$  and  $S \subset K$ . We say that a cut  $(U, V \setminus U)$  is  $S$ -separating if it separates the terminal subset  $S$  from its complement  $K \setminus S$ , i.e.,  $U \cap K$  is either  $S$  or  $K \setminus S$ . We will refer to such cut as a *terminal cut*. The cutset  $\delta(U)$  of a cut  $(U, V \setminus U)$  represents the edges that have one endpoint in  $U$  and the other one in  $V \setminus U$ . The cost  $\text{cap}_G(\delta(U))$  of a cut  $(U, V \setminus U)$  is the sum over all capacities of the edges belonging to the cutset. We let  $\text{mincut}_G(S, K \setminus S)$  denote the  $S$ -separating cut of minimum cost in  $G$ . A graph  $H = (V_H, E_H, c_H)$ ,  $K \subset V_H$  is a *vertex cut sparsifier* of  $G$  with *quality*  $q \geq 1$  if for any  $S \subset K$ ,  $\text{mincut}_G(S, K \setminus S) \leq \text{mincut}_H(S, K \setminus S) \leq q \cdot \text{mincut}_G(S, K \setminus S)$ .

Let  $d$  be a demand function over terminal pairs in  $G$  such that  $d(x, x') = d(x', x)$  and  $d(x, x) = 0$  for all  $x, x' \in K$ . We denote by  $P_{xx'}$  the set of all paths between vertices  $x$  and  $x'$ , for all  $x, x' \in K$ . Further, let  $P_e$  be the set of all paths using edge  $e$ , for all  $e \in E$ . A *concurrent (multi-commodity)* flow  $f$  of *throughput*  $\lambda$  is a function over terminal paths in  $G$  such that (1)  $\sum_{p \in P_{xx'}} f(p) \geq \lambda d(x, x')$ ,

for all distinct terminal pairs  $x, x' \in K$  and (2)  $\sum_{p \in P_e} f(p) \leq c(e)$ , for all  $e \in E$ . We let  $\lambda_G(d)$  denote the *throughput of the concurrent flow* in  $G$  that attains the largest throughput and we call a flow achieving this throughput the *maximum concurrent flow*. A graph  $H = (V_H, E_H, c_H)$ ,  $K \subset V_H$  is a *vertex flow sparsifier* of  $G$  with *quality*  $q \geq 1$  if for every demand function  $d$ ,  $\lambda_G(d) \leq \lambda_H(d) \leq q \cdot \lambda_H(d)$ .

Let  $G = (V, E, \ell)$  with  $K \subset V$  be a  $k$ -terminal graph, where we replace the capacity function  $c$  with a length function  $\ell : E \rightarrow \mathbb{R}_{\geq 0}$ . For a terminal pair  $(x, x') \in K$ , let  $d_G(x, x')$  denote the shortest path with respect to the edges lengths  $\ell$  in  $G$ . A graph  $H = (V', E', \ell')$  is an *vertex distance sparsifier* of  $G$  with *quality* or *stretch*  $\alpha \geq 1$  if for any  $x, x' \in K$ ,  $d_G(x, x') \leq d_H(x, x') \leq \alpha \cdot d_G(x, x')$ .

## 2.1 Wye-Delta Transformations

In this section we investigate the applicability of some graph reduction techniques that aim at reducing the number of non-terminals in a  $k$ -terminal graph. Khan and Raghavendra [22] were the first to apply such techniques in the context of Vertex Cut Sparsification for tree networks.

We start by reviewing the so-called *Wye-Delta* operations in graph reductions. These operations consist of five basic rules, which we describe below. (See Fig. 1 for illustrations.)

1. *Degree-one reduction*: Delete a degree-one non-terminal and its incident edge.
2. *Series reduction*: Delete a degree-two non-terminal  $y$  and its incident edges  $(x, y)$  and  $(y, z)$ , and add a new edge  $(x, z)$  of capacity  $\min\{c(x, y), c(y, z)\}$ .
3. *Parallel reduction*: Replace all parallel edges by a single edge whose capacity is the sum over all capacities of parallel edges.
4. *Wye-Delta transformation*: Let  $x$  be a degree-three non-terminal with neighbours  $\delta(x) = \{u, v, w\}$ . Assume w.l.o.g.<sup>1</sup> that for any pair  $(u, v) \in \delta(x)$ ,  $c(u, x) + c(v, x) \geq c(w, x)$ , where  $w \in \delta(v) \setminus \{u, v\}$ . Then we can delete  $x$  (along with all its incident edges) and add edges  $(u, v)$ ,  $(v, w)$  and  $(w, u)$  with capacities  $(c(u, x) + c(v, x) - c(w, x))/2$ ,  $(c(v, x) + c(w, x) - c(u, x))/2$  and  $(c(u, x) + c(w, x) - c(v, x))/2$ , respectively.
5. *Delta-Wye transformation*: Delete the edges of a triangle connecting  $x, y$  and  $z$ , introduce a new non-terminal vertex  $w$  and add new edges  $(w, x)$ ,  $(w, y)$  and  $(w, z)$  with edge capacities  $c(x, y) + c(x, z)$ ,  $c(x, y) + c(y, z)$  and  $c(x, z) + c(y, z)$  respectively.

The following lemma (which follows from the above definitions) shows that the above rules preserve exactly all terminal minimum cuts.

**Lemma 2.1.** *Let  $G$  be a  $k$ -terminal graph and  $G'$  be a  $k$ -terminal graph obtained from  $G$  by applying one of the rules 1 – 5. Then  $G'$  is a quality 1-vertex cut sparsifier of  $G$ .*

For our application, it will be useful to enrich the set of rules by introducing two new operations. These operations can be realized as series of the operations 1-5. (See Fig. 2 and 3 for illustrations.)

6. *Edge deletion (with vertex  $x$ )*: If a degree-three non-terminal has neighbours  $u, v$ , the edge  $(u, v)$  can be deleted, if it exists. To achieve this, we use a Delta-Wye transformation followed by a series reduction.

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<sup>1</sup>Suppose there exist a pair  $(u, v) \in \delta(x)$  with  $c(u, x) + c(v, x) < c(w, x)$ , where  $w \in \delta(v) \setminus \{u, v\}$ . Then we can simply set  $c(w, x) = c(u, x) + c(v, x)$ , since any terminal minimum cut would cut the edges  $(u, x)$  and  $(v, x)$  instead of the edge  $(w, x)$ .

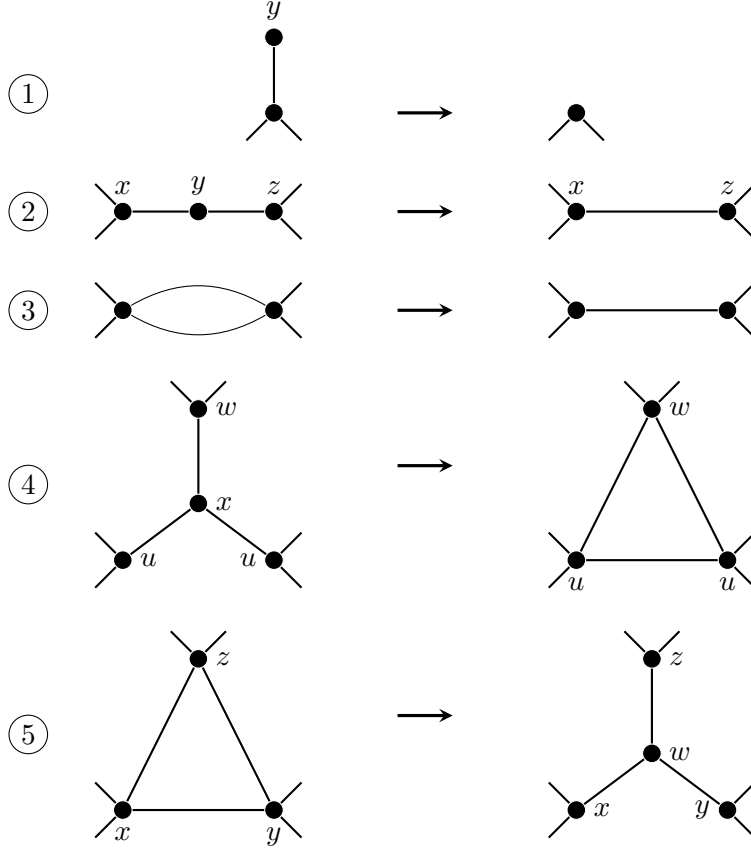


Figure 1: Wye-Delta operations: 1. Degree-one reduction; 2. Series reduction; 3. Parallel reduction; 4. Wye-Delta transformation; 5. Delta-Wye transformation.

7. *Edge replacement*: If a degree-four non-terminal vertex has neighbours  $x, u, v, w$ , then if the edge  $(x, u)$  exists, it can be replaced by the edge  $(v, w)$ . To achieve this, we use a Delta-Wye transformation followed by a Wye-Delta transformation.

A  $k$ -terminal graph  $G$  is *Wye-Delta* reducible to another  $k$ -terminal graph  $H$ , if  $G$  is reduced to  $H$  by repeatedly applying one of the operations 1-7. We obtain the following:

**Lemma 2.2.** *Let  $G$  and  $H$  be  $k$ -terminal graphs. Moreover, let  $G$  be Wye-Delta reducible to  $H$ . Then  $H$  is a quality 1-vertex cut sparsifier of  $G$ .*

*Proof.* Observe that the rules 1-7 do not affect any terminal vertex and each rule preserves exactly all terminal minimum cuts by Lemma 2.1. An induction on the number of rules needed to reduce  $G$  to  $H$  proves the claim.  $\square$

## 2.2 Graph Embeddings

Throughout this paper, we will be dealing with the embedding of a planar graph into a square *grid* graph. One way of drawing graphs in the plane are *orthogonal grid-embeddings* [37]. In such setting, the vertices correspond to distinct points and edges consist of alternating sequences of vertical and horizontal segments. Equivalently, one can view this as drawing our input graph as

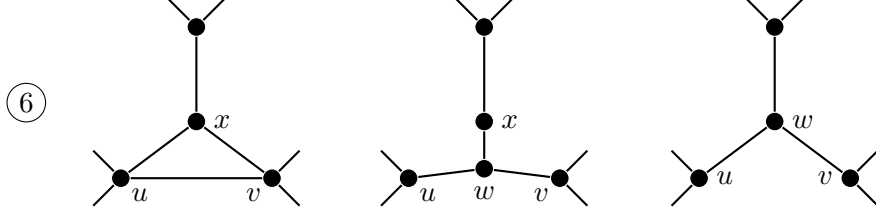


Figure 2: Edge deletion transformation. Edge capacities are omitted.

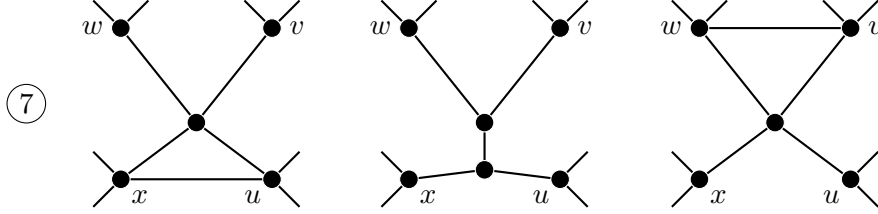


Figure 3: Edge replacement transformation. Edge capacities are omitted.

a subgraph of some grid. Formally, a *node-embedding*  $\rho$  of  $G_1 = (V_1, E_1)$  into  $G_2 = (V_2, E_2)$  is a one-to-one mapping that maps  $V_1$  into  $V_2$ , and  $E_1$  into paths in  $G_2$ , i.e.,  $(u, v)$  maps to a path from  $\rho(u)$  to  $\rho(v)$ , such that every pair of paths that correspond to two different edges in  $G_1$  is vertex-disjoint (except possibly at the endpoints). If  $G_2$  is a planar graph, then  $\rho(G_1)$  and  $G_1$  are also planar. Thus, if  $G_1$  and  $G_2$  are planar we then refer to  $\rho$  as an *orthogonal embedding*. (See Fig. 4 for an example.) Moreover, given a planar graph  $G_1$  drawn in the plane, the embedding  $\rho$  is called *region-preserving* if  $\rho(G_1)$  and  $G_1$  have the same planar topological embedding.

Let  $G_1$  be a  $k$ -terminal graph. Since the embedding does not affect the vertices of  $G_1$ , the terminals of  $G_1$  are also terminals in  $\rho(G_1)$ . Although the embedding does not consider capacity of the edges in  $G_1$ , we can still argue that such an embedding preserves all terminal minimum cuts. We make use of the following operation:

1. *Edge subdivision:* Let  $(u, v)$  be an edge of capacity  $c(u, v)$ . Delete  $(u, v)$ , introduce a new vertex  $w$  and add edges  $(u, w)$  and  $(w, v)$ , each of capacity  $c(u, v)$ .

**Lemma 2.3.** *Let  $\rho$  be a node-embedding and let  $G_1$  and  $\rho(G_1)$  be  $k$ -terminal graphs defined as above. Then  $\rho(G_1)$  preserves exactly all terminal minimum cuts of  $G$ .*

*Proof.* We can view each path obtained from the embedding as taking the edge corresponding to the path endpoints in  $G_1$  and performing edge subdivisions finitely many times. We claim that such subdivisions preserve all terminal cuts.

Indeed, let us consider a single edge subdivision for  $(u, v)$  (the general claim then follows by induction on the number of edge subdivisions). Fix  $S \subset K$  and consider some  $S$ -separating minimum cut  $(U, V \setminus U)$  in  $G_1$  cutting  $(u, v)$ . Then, in the transformed graph  $\rho(G_1)$ , we can simply cut either the edge  $(u, w)$  or  $(w, v)$ . Since by construction, the new edge has the same capacity as the subdivided edge, we get that  $\text{cap}_{\rho(G_1)}(\delta(U)) = \text{cap}_{G_1}(\delta(U))$ , and in particular  $\text{mincut}_{\rho(G_1)}(S, K \setminus S) \leq \text{mincut}_{G_1}(S, K \setminus S)$ .

Furthermore, since  $G_1$  is obtained by contracting two edges of the same capacity of  $\rho(G_1)$ , for



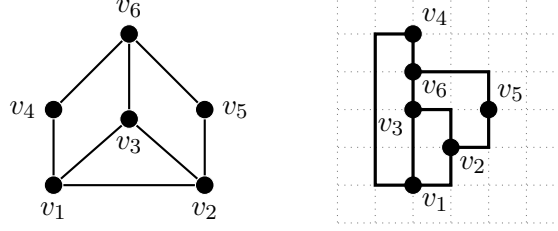


Figure 4: Orthogonal graph embedding into grid.

any  $S$ -separating minimum cut  $(U, V \setminus U)$  in  $\rho(G_1)$ , we have  $\text{cap}_{\rho(G_1)}(\delta(U)) \geq \text{cap}_{G_1}(\delta(U))$ , and in particular  $\text{mincut}_{\rho(G_1)}(S, K \setminus S) \geq \text{mincut}_{G_1}(S, K \setminus S)$ . Combining the above gives the lemma.  $\square$

### 3 An Exact Vertex Cut Sparsifier of size $O(k^2)$

In this section we show that given a  $k$ -terminal planar graph, where all terminals lie on the same face, one can construct a quality 1-vertex cut sparsifier of size  $O(k^2)$ . Note that it suffices to consider the case when all terminals lie on the *outer* face. Our approach closely follows the work of Gitler [17] in the context of Wye-Delta reducibility of graphs, which does not consider preservation of terminal cuts. Apart from bringing his work to the vertex sparsification community, we also fill in some gaps and provide the necessary details for applying his technique to the cut setting.

#### 3.1 Embedding into Grids

It is well-known that one can obtain an orthogonal embedding of a planar graph with maximum-degree at most three into a grid (see Valiant [37]). However, our input planar graph can have arbitrary large maximum-degree. In order to be able to make use of such an embedding, we need to first reduce our input graph to a bounded-degree graph while preserving planarity and all terminal minimum cuts. We achieve this by making use of a *vertex splitting* technique, which we describe below.

Given a  $k$ -terminal planar graph  $G' = (V', E', c')$  with  $K \subset V'$  lying on the outer face, vertex splitting produces a  $k$ -terminal planar graph  $G = (V, E, c)$  with  $K \subset V$  such that the maximum degree of  $G$  is at most three. Specifically, for each vertex  $v$  of degree  $d > 3$  with neighboring vertices  $u_1, \dots, u_d$ , we delete  $v$  and introduce new vertices  $v_1, \dots, v_d$  along with edges  $\{(v_i, v_{i+1}) : i = 1, \dots, d-1\}$ , each of capacity  $C + 1$ , where  $C = \sum_{e \in E'} c'(e)$ . Further, we replace the edges  $\{(u_i, v) : i = 1, \dots, d\}$  with  $\{(u_i, v_i) : i = 1, \dots, d\}$ , each of corresponding capacity. If  $v$  is a terminal vertex, we set one of the  $v_i$ 's to be a terminal vertex. It follows that the resulting graph  $G$  is planar and terminals can be still embedded on the outer face. Note that while the degree of every vertex  $v_i$  is at most 3, the degree of any other vertex is not affected.

**Claim 3.1.** *Let  $G'$  and  $G$  be  $k$ -terminal graphs defined as above. Then  $G$  preserves exactly all minimum terminal cuts of  $G'$ , i.e.,  $G$  is a quality-1 cut sparsifier of  $G'$ .*

*Proof.* It suffices to prove the case where  $G$  is obtained from  $G'$  by a single vertex splitting. Then the claim follows by induction on the number of vertex splittings required to transform  $G'$  to  $G$ .

Let  $S \subset K$  and  $(U, V \setminus U)$  be an  $S$ -separating cut in  $G$  of size  $\text{mincut}_G(S, K \setminus S)$ . Suppose towards contradiction that  $\delta(U)$  contains an edge of the form  $(v_j, v_{j+1})$ , for some  $j$ , which in turn gives that  $\text{cap}(\delta(U)) \geq C + 1$ . Then we can move all the points  $v_i$  to one of the sides of the cut

$(U, V \setminus S)$  and obtain a new  $S$ -separating cut in  $G$  of cost at most  $C$ , contradicting the fact that  $(U, V \setminus U)$  is a minimum terminal cut. Hence, it follows that  $\delta(U)$  uses either edges that are in both  $G$  and  $G'$  or edges of the form  $(u_i, v_i)$ , which by construction have the same capacity as the edges  $(u_i, v)$  in  $G'$ . Thus, an  $S$ -separating minimum cut in  $G$  corresponds to an  $S$ -separating minimum cut in  $G'$  of the same cost. Since  $S$  was chosen arbitrarily, the claim follows.  $\square$

Let  $G = (V, E)$  be a  $k$ -terminal graph obtained by vertex splitting of all vertices of degree larger than 3 of  $G' = (V', E')$ . Further, let  $n' = |V'|$ ,  $m' = |E'|$ ,  $n = |V|$  and  $m = |E|$ . Then it is easy to show that  $n \leq 2m'$  and  $m \leq m' + n \leq 3m'$ . Since  $G'$  is planar, we have that  $n = O(n')$  and  $m = O(n')$ . Thus, by just a linear blow-up on the size of vertex and edge sets, we may assume w.l.o.g. that our input graph is a planar graph of degree at most three.

Valiant [37] and Tamassia et al. [35] showed that a  $k$ -terminal planar graph  $G$  with  $n$  vertices and degree at most three admits an orthogonal region-preserving embedding into some square grid of size  $O(n) \times O(n)$ . By Lemma 2.3, we know that the resulting graph exactly preserves all terminal minimum cuts of  $G$ . We remark that since the embedding is region-preserving, the outer face of the input graph is embedded to the outer face of the grid. Therefore, all terminals in the embedded graph lie on the outer face of the grid. Performing appropriate edge subdivisions, we can make all the terminals lie on the boundary of some possibly larger grid. Further, we can add dummy non-terminals and zero edge capacities to transform our graph into a full-grid  $H$ . We observe that the latter does not affect any terminal min-cut. The above leads to the following:

**Lemma 3.2.** *Given a  $k$ -terminal planar graph  $G$ , where all terminals lie on the outer face, there exists a  $k$ -terminal grid graph  $H$ , where all terminals lie on the boundary such that  $H$  preserves exactly all terminal minimum cuts of  $G$ . The resulting graph has  $O(n^2)$  vertices and edges.*

### 3.2 Embedding Grids into Half-Grids

Next, we show how to embed square grids into half-grid graphs (see Section 2), which will facilitate the application of Wye-Delta transformations. The existence of such an embedding was claimed in the work of Gitler [17], but no details on its construction were given.

Let  $G$  be a  $k$ -terminal square grid on  $n \times n$  vertices where terminals lie on the boundary of the grid. We obtain the following:

**Lemma 3.3.** *There exists a node embedding of the grid  $G$  into  $T_k^\ell$ , where  $\ell = 4n - 3$ .*

*Proof.* Our construction works as follows (See Fig. 5 for an example). We first fix an ordering on the vertices lying on the boundary of the grid in the order induced by the grid. Then we embed each vertex according to that order into the diagonal vertices of the half-grid, along with the edges that form the boundary of the grid. The sub-grid obtained by removing all boundary vertices is embedded appropriately into the upper-part of the half-grid. Finally, we show how to embed edges between the boundary and the sub-grid vertices and argue that such an embedding is indeed vertex-disjoint for any pair of paths.

We start with the embedding of the vertices of  $G$ . Let us first consider the boundary vertices. The ordering imposed on these vertices can be viewed as starting with the upper-right vertex  $(1, n)$  and visiting the rest of vertices in a counter-clockwise direction until reaching the vertex  $(2, n)$ . We map the vertices on the boundary as follows.

1. The vertex  $(1, j)$  is mapped to the vertex  $(n - j + 1, n - j + 1)$  for  $j = 2, \dots, n$ ,
2. The vertex  $(i, 1)$  is mapped to the vertex  $(n + i - 1, n + i - 1)$  for  $i = 1, \dots, n - 1$ ,

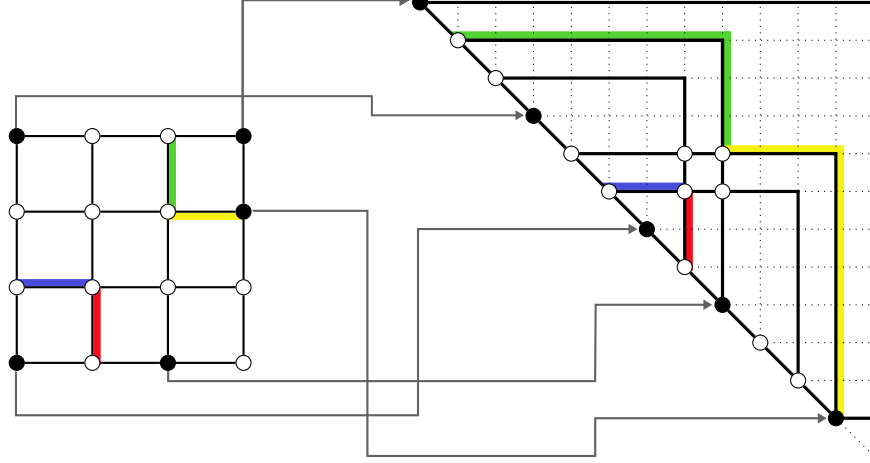


Figure 5: Embedding grid into half-grid. Black vertices represent terminals while white vertices represent non-terminals. The counter-clockwise ordering starts at the top right terminal. Coloured edges and paths correspond to the mapping of the respective edges: blue for edges  $((i, 1), (i, 2))$ , red for edges  $((n-1, j), (n, j))$ , green for edges  $((1, j), (2, j))$  and yellow for edges  $((i, n-1), (i, n))$ , where  $i, j = 2, \dots, n-1$ .

3. The vertex  $(n, j)$  is mapped to the vertex  $(2n + j - 2, 2n + j - 2)$  for  $j = 1, \dots, n-1$ ,
4. The vertex  $(i, n)$  is mapped to the vertex  $(4n - i - 2, 4n - i - 2)$  for  $i = 2, \dots, n$ .

Now we consider the vertices that belong to the induced sub-grid  $S$  of  $G$  of size  $(n-2)^2$  when removing the boundary vertices of our input grid. We map the vertex  $(i, j)$  to the vertex  $(n + i - 1, 2n + j - 2)$  for  $i, j = 2, \dots, n-1$ . In other words, for every vertex of  $S$  we make a vertical shift by  $n-1$  units and an horizontal shift by  $2n-2$  units. By construction, it is not hard to check that every vertex of  $G$  is mapped to a different vertex of  $T_k^\ell$  and all terminal vertices lie on the diagonal of  $T_k^\ell$ .

We continue with the embedding of the edges of  $G$ . First, every edge between two boundary vertices in  $G$  is embedded to the edge between the corresponding mapped diagonal vertices of  $T_k^\ell$ , except the edge between  $(1, n)$  and  $(2, n)$ . For this edge, we define an edge embedding between the corresponding vertices  $(1, 1)$  and  $(4n-4, 4n-4)$  of  $T_k^\ell$  by using the path:

$$(1, 1) \rightarrow (1, 2) \rightarrow \dots \rightarrow (1, 4n-3) \rightarrow (2, 4n-3) \rightarrow \dots \rightarrow (4n-4, 4n-3) \rightarrow (4n-4, 4n-4).$$

Next, every edge of the sub-grid  $S$  is embedded in to the edge connecting the mapped endpoints of that edge in  $T_k^\ell$ . In other words, if  $(i, j)$  and  $(i', j')$  were connected by an edge  $e$  in  $S$ , then  $(n + i - 1, 2n + j - 2)$  and  $(n + i' - 1, 2n + j' - 2)$  are connected by an edge  $e'$  in  $T_k^\ell$  and  $e$  is mapped to  $e'$ . Finally, the only edges that remain are those connecting a boundary vertex of  $G$  with a boundary vertex of  $S$ . We distinguish four cases depending on the edge position.

1. The edge  $((i, 2), (i, 1))$  is mapped to the horizontal path given by:

$$(n + i - 1, 2n) \rightarrow (n + i - 1, 2n - 1) \rightarrow \dots \rightarrow (n + i - 1, n + i - 1) \text{ for } i = 2, \dots, n-1.$$

2. The edge  $((n-1, j), (n, j))$  is mapped to the vertical path given by:

$$(2n-2, 2n+j-2) \rightarrow (2n-1, 2n+j-2) \rightarrow \dots \rightarrow (2n+j-2, 2n+j-2) \text{ for } j = 2, \dots, n-1.$$

3. The edge  $((2, j), (1, j))$  is mapped to the  $L$ -shaped path:

$$(n+1, 2n+j-2) \rightarrow (n, 2n+j-2) \rightarrow \dots \rightarrow (n-j+1, 2n+j-2) \\ \rightarrow (n-j+1, 2n+j-3) \rightarrow \dots \rightarrow (n-j+1, n-j+1) \text{ for } j = 2, \dots, n-1.$$

4. The edge  $((i, n-1), (i, n))$  is mapped to the  $L$ -shaped path:

$$(n+i-1, 3n-3) \rightarrow (n+i-1, 3n-2) \rightarrow \dots \rightarrow (n+i-1, 4n-i-2) \\ \rightarrow (n+i, 4n-i-2) \rightarrow \dots \rightarrow (4n-i-2, 4n-i-2) \text{ for } i = 2, \dots, n-1.$$

By construction, it follows that the paths in our edge embedding are vertex disjoint.  $\square$

### 3.3 Reducing Half-Grids

We now review the construction of Gitler [17], which shows how to reduce half-grids to much smaller half-grids (excluding diagonal edges) whose size depends only on  $k$ . For the sake of completeness, we provide here a full proof. Recall that  $\hat{T}_k^n$  is the graph  $T_k^n$  without the diagonal edges.

**Lemma 3.4** ([17]). *For any positive integers  $k, n$  with  $k < n$ ,  $T_k^n$  is Wye-Delta reducible to  $\hat{T}_k^k$ .*

*Proof.* For sake of simplicity, we assume w.l.o.g that the four vertices  $(1, 1)$ ,  $(2, 2)$ ,  $(n-1, n-1)$  and  $(n, n)$  are terminals<sup>2</sup>. Furthermore, we say that two terminals  $(i, i)$  and  $(j, j)$  are *adjacent* iff  $i < j$  and there is no terminal  $(\ell, \ell)$  such that  $i < \ell < j$ .

We next describe the reduction procedure. Also see Fig. 6 for an example. The reduction procedure starts by removing the diagonal edges of  $T_k^n$ , thus producing the graph  $\hat{T}_k^n$ . Specifically, the two edges  $((1, 1), (2, 2))$  and  $((n-1, n-1), (n, n))$  are removed using an edge deletion operation. For each remaining diagonal edge of the form  $((i, i), (i+1, i+1))$ ,  $i = 2, \dots, n-2$  we repeatedly apply an edge replacement operation until the edge is incident to a boundary vertex  $(1, j)$  or  $(j, n)$  of the grid, where an edge deletion operation with one of the neighbours of  $(1, j)$  resp.  $(j, n)$  as vertex  $x$  is applied.

Now, we know that all non-terminals of the form  $(i, i)$  are degree-two vertices, thus a series reduction is applied on each of them. This produces new diagonal edges, which are effectively reduced by the above procedure. We keep removing the newly-created degree-two non-terminal vertices and the newly-created edges until no further removals are possible. At this point, the only degree-2 vertices are terminal vertices.

The resulting graph has a staircase structure, where for every pair of adjacent terminals  $(i, i)$  and  $(j, j)$ , there is a non-terminal  $(i, j)$  of degree three or four, namely, the intersection vertex, and a (possibly empty) sequence of degree-three non-terminals that lie on the boundary path from  $(i, i)$  to  $(j, j)$ . For  $k = i+1, \dots, j-1$ , let  $(i, k)$  and  $(k, j)$  be the degree-three non-terminals lying on the row and the column subpath, respectively. Additionally, for  $k = i+1, \dots, j-1$ , let  $C_k^i = \{(i', k) : i' = i, \dots, 1\}$ , resp.  $R_k^j = \{(k, j') : j' = j, \dots, n\}$  be the vertices sharing the same column, resp. row with  $(i, k)$ , resp.  $(k, j)$ . We next show that the vertices belonging to  $C_k^i$  and  $R_k^j$  can be removed.

The removal process works as follows. For  $k = i+1, \dots, j-1$ , we start by choosing a degree 3 vertex  $(i, k)$  and its corresponding column  $C_k^i$ . Then we apply a Wye-Delta transformation on  $(i, k)$ , thus creating two new diagonal edges. Similarly as above, we remove such edges by repeatedly

<sup>2</sup>If they are not terminals, we can simply define them as terminals, thus increasing the number of terminals to  $k+4 = O(k)$ .

applying an edge replacement operation until they have been pushed to the boundary of the grid, where an edge deletion operation is applied. In the resulting graph, the vertex  $(i-1, k) \in C_k^i$  is now a degree-three non-terminal. We apply the same procedure to this vertex. Applying such a procedure to all remaining vertices of  $C_k^i$ , we eliminate a column of the grid. Symmetrically, the same process applies to the case when we want to remove the row  $R_k^j$  corresponding to the vertex  $(k, j)$ .

Applying the above removal process for every adjacent terminal pair and the corresponding degree-three non-terminals, we end up with the graph  $\hat{T}_k^k$ , where every diagonal vertex is a terminal. By definition, it follows that  $\hat{T}_k^k$  has at most  $O(k^2)$  vertices.  $\square$

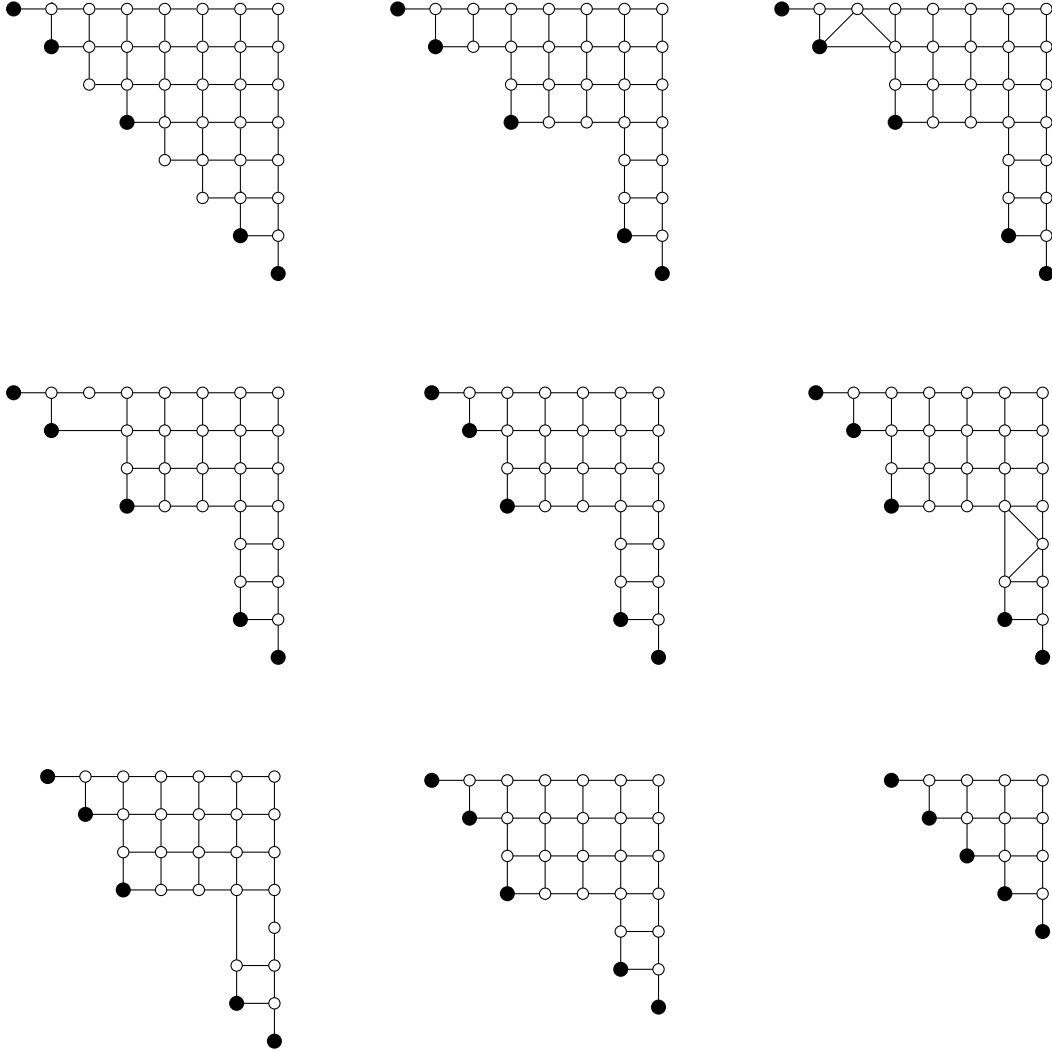


Figure 6: Half-Grid Reduction.

### 3.4 Bringing the Pieces Together

Combining the above reductions leads to the following theorem:

**Theorem 3.5.** *Let  $G$  be a  $k$ -terminal planar graph where all terminals lie on the outer face. Then  $G$  admits a quality 1-vertex cut sparsifier of size  $O(k^2)$ , which is also a planar graph.*

*Proof.* Let  $n$  denote the number of vertices in  $G$ . First, we apply Lemma 3.2 on  $G$  to obtain a grid graph  $H$  with  $O(n^2)$  vertices, which preserves exactly all terminal minimum cuts of  $G$ . We then apply Lemma 3.3 on  $H$  to obtain a node embedding  $\rho$  into the half-grid  $T_k^\ell$ , where  $\ell = 4n - 3$ . By Lemma 2.3,  $\rho(H)$  preserves exactly all terminal minimum cuts of  $H$ . We can further extend  $\rho(H)$  to the full half-grid  $T_k^\ell$ , if dummy non-terminals and zero edge capacities are added. Finally, we apply Lemma 3.4 on  $T_k^\ell$  to obtain a Wye-Delta reduction to the reduced half-grid graph  $\hat{T}_k^k$ . It follows by Lemma 2.2 that  $\hat{T}_k^k$  is a quality 1-vertex cut sparsifier of  $T_k^\ell$ , where the size guarantee is immediate from the definition of  $\hat{T}_k^k$ .  $\square$

## 4 Extension to Flow Sparsifiers

In this section we show that given a  $k$ -terminal planar graph, where all terminals lie on the outer face, one can construct a quality 1 vertex flow sparsifier of size  $O(k^2)$ . Our result follows from combining the observation of Andoni et al. [3] for constructing flow-sparsifiers using flow/cut gaps and the flow/cut gap result of Okamura and Seymour [33].

Given a  $k$ -terminal graph and a demand function  $d$ , recall that  $\lambda_G(d)$  is the maximum fraction of  $d$  that can be routed in  $G$ . We define the *sparsity* of a cut  $(U, V \setminus U)$  to be

$$\Phi_G(U, d) := \frac{\text{cap}(\delta(U))}{\sum_{i,j: \{i,j\} \cap U = 1} d_{ij}}$$

and the *sparsest cut* as  $\Phi_G(d) := \min_{U \subset V} \Phi_G(U, d)$ . Then the *flow-cut gap* is given by

$$\gamma(G) := \max\{\Phi_G(d)/\lambda_G(d) : d \in \mathbb{R}_+^{\binom{k}{2}}\}.$$

We will make use of the following theorem:

**Theorem 4.1** ([3]). *Given a  $k$ -terminal graph  $G$  with terminals  $K$ , let  $G'$  be a quality  $\beta \geq 1$  vertex cut sparsifier for  $G$ . Then for every demand function  $d \in \mathbb{R}_+^{\binom{k}{2}}$ ,*

$$\frac{1}{\gamma(G')} \leq \frac{\lambda_{G'}(d)}{\lambda_G(d)} \leq \beta \cdot \gamma(G).$$

*Therefore, the graph  $G'$  with edge capacities scaled up by  $\gamma(G')$  is a quality  $\beta \cdot \gamma(G) \cdot \gamma(G')$  vertex flow sparsifier of size  $|V(G')|$  for  $G$ .*

This leads to the following corollary:

**Corollary 4.2.** *Let  $G$  be a  $k$ -terminal planar graph where all terminals lie on the outer face. Then  $G$  admits a quality 1-vertex flow sparsifier of size  $O(k^2)$ .*

*Proof.* Given a  $k$ -terminal planar graph where all terminals lie on the outer face, Theorem 3.5 shows how to construct a vertex cut sparsifier  $G'$  with quality  $\beta = 1$  and size  $O(k^2)$ , which is also a planar graph with all the  $k$  terminals lying on the outer face. Okamura and Seymour [33] showed that for every  $k$ -terminal planar graph  $G$  with terminals lying on the outer face the flow-cut gap is 1. This implies that  $\gamma(G) = 1$  and  $\gamma(G') = 1$ . Invoking Theorem 4.1 we get that  $G'$  is a quality 1-vertex flow sparsifier of size  $O(k^2)$  for  $G$ .  $\square$

## 5 Planar Exact Distance Sparsifiers

We next argue that a symmetric approach applies to the construction of vertex sparsifiers that preserve distances. Concretely, we prove that given a  $k$ -terminal planar graph, where all terminals lie on the outer face, one can construct a stretch 1-vertex distance sparsifier of size  $O(k^2)$ , which is also a planar graph. It is not hard to see that almost all arguments that we used about vertex cut sparsifiers go through, except some adaptations regarding edge lengths in the Wye-Delta rules, edge subdivision operation and vertex splitting operation.

We start adapting the Wye-Delta operations.

1. *Degree-one reduction:* Delete a degree-one non-terminal and its incident edge.
2. *Series reduction:* Delete a degree-two non-terminal  $y$  and its incident edges  $(x, y)$  and  $(y, z)$ , and add a new edge  $(x, z)$  of length  $\ell(x, y) + \ell(y, z)$ .
3. *Parallel reduction:* Replace all parallel edges by a single edge whose length is the minimum over all lengths of parallel edges.
4. *Wye-Delta transformation:* Let  $x$  be a degree-three non-terminal with neighbours  $\delta(x) = \{u, v, w\}$ . Delete  $x$  (along with all its incident edges) and add edges  $(u, v)$ ,  $(v, w)$  and  $(w, u)$  with lengths  $\ell(u, x) + \ell(v, x)$ ,  $\ell(v, x) + \ell(w, x)$  and  $\ell(w, x) + \ell(u, x)$ , respectively.
5. *Delta-Wye transformation:* Let  $x, y$  and  $z$  be the vertices of the triangle connecting them. Assume w.l.o.g.<sup>3</sup> that for any triangle edge  $(x, y)$ ,  $\ell(x, y) \leq \ell(x, z) + \ell(y, z)$ , where  $z$  is the other triangle vertex. Delete the edges of the triangle, introduce a new vertex  $w$  and add new edges  $(w, x)$ ,  $(w, y)$  and  $(w, z)$  with edge lengths  $(\ell(x, y) + \ell(x, z) - \ell(y, z))/2$ ,  $(\ell(x, z) + \ell(y, z) - \ell(x, y))/2$  and  $(\ell(x, y) + \ell(y, z) - \ell(x, z))/2$ , respectively.

The following lemma shows that the above rules preserve exactly all shortest path distances between terminal pairs.

**Lemma 5.1.** *Let  $G$  be a  $k$ -terminal graph and  $G'$  be a  $k$ -terminal graph obtained from  $G$  by applying one of the rules 1-5. Then  $G'$  is a quality 1-vertex distance sparsifier of  $G$ .*

We remark that there is no need to re-define the Edge deletion and replacement operations, since they are just a combination of the above rules. An analogue of Lemma 2.2 can also be shown for distances.

We now modify the Edge subdivision operation, which is used when dealing with graph embeddings (see Section 2.2).

1. *Edge subdivision:* Let  $(u, v)$  be an edge of length  $\ell(u, v)$ . Delete  $(u, v)$ , introduce a new vertex  $w$  and add edges  $(u, w)$  and  $(w, v)$ , each of length  $\ell(u, v)/2$ .

We now prove an analogue to Lemma 2.3.

**Lemma 5.2.** *Let  $\rho$  be a node embedding and let  $G_1$  and  $\rho(G_1)$  be  $k$ -terminal graphs as defined in Section 2.2. Then  $\rho(G_1)$  preserves exactly all shortest path distances between terminal pairs.*

---

<sup>3</sup>Suppose there exists a triangle edge  $(x, y)$  with  $\ell(x, y) > \ell(x, z) + \ell(y, z)$ , where  $z$  is the other triangle vertex. Then we can simply set  $\ell(x, y) = \ell(x, z) + \ell(y, z)$ , since any shortest path between terminal pairs would use the edges  $(x, z)$  and  $(y, z)$  instead of the edge  $(x, y)$ .

*Proof.* We can view each path obtained from the embedding as taking the edge corresponding to that path endpoints in  $G_1$  and performing edge subdivisions finitely many times. We claim that such subdivisions preserve all terminal shortest paths.

Indeed, let us consider a single edge subdivision for  $(u, v)$  (the general claim then follows by induction on the number of edge subdivisions). Fix  $x, x' \in K$  and consider some shortest path  $p(x, x')$  in  $G_1$  that uses  $(u, v)$ . We can construct in  $\rho(G_1)$  a path  $q(x, x')$  of the same length as follows: traverse the subpath  $p(x, u)$ , traverse the edges  $(u, w)$  and  $(w, v)$  and finally traverse the subpath  $p(v, x')$ . It follows that  $\sum_{e \in p(x, x')} \ell(e) = \sum_{e \in q(x, x')} \ell(e)$ , and thus  $d_{\rho(G_1)}(s, t) \leq d_{G_1}(s, t)$ .

On the other hand, fix  $x, x' \in K$  and consider some shortest path  $p'(x, x')$  in  $\rho(G_1)$  that uses the two subdivided edges  $(u, w)$  and  $(w, v)$  (note that it cannot use only one of them). We can construct in  $G_1$  a path  $q'(x, x')$  of the same length as follows: traverse the subpath  $p'(x, u)$ , traverse the edge  $(u, v)$  and finally traverse the subpath  $p'(v, x')$ . It follows that  $\sum_{e \in p'(x, x')} \ell(e) = \sum_{e \in q'(x, x')} \ell(e)$  and thus  $d_{G_1}(s, t) \leq d_{\rho(G_1)}(s, t)$ . Combining the above gives the lemma.  $\square$

We next consider vertex splitting for graphs whose maximum degree is larger than three. For each vertex  $v$  of degree  $d > 3$  with  $u_1, \dots, u_d$  adjacent to  $v$ , we delete  $v$  and introduce new vertices  $v_1, \dots, v_d$  along with edges  $\{(v_i, v_{i+1}) : i = 1, \dots, d-1\}$ , each of length 0. Furthermore, we replace the edges  $\{(u_i, v) : i = 1, \dots, d\}$  with  $\{(u_i, v_i) : i = 1, \dots, d\}$ , each of corresponding length. If  $v$  is a terminal vertex, we make one of the  $v_i$ 's be a terminal vertex. An analogue to Claim 3.1 gives that the resulting graph preserves all terminal shortest path distances.

We finally note that whenever we add dummy edges of capacity 0 in the cut setting, we replace them by edges of length  $D + 1$  in the distance setting, where  $D$  is the sum over all edge lengths in the graph we consider. Since any shortest path in the graph does not use the added edges, the terminal shortest path remain unaffected.

The above discussion leads to the following theorem.

**Theorem 5.3.** *Let  $G$  be a  $k$ -terminal planar graph where all terminals lie on the outer face. Then  $G$  admits a stretch 1-vertex distance sparsifier of size  $O(k^2)$ , which is also a planar graph.*

## 6 Incompressibility of distances in $k$ -terminal graphs

In this section we prove the following incompressibility result (i.e., Theorem 1.2) concerning the trade-off between quality and size of any compression function when estimating terminal distances in  $k$ -terminal graphs: for every  $\varepsilon > 0$  and  $t \geq 2$ , there exists a (sparse)  $k$ -terminal  $n$ -vertex graph such that  $k = o(n)$ , and that any compression algorithm that approximates pairwise terminal distances within a factor of  $t - \varepsilon$  or an additive error  $2t - 3$  must use  $\Omega(k^{1+1/(t-1)})$  bits. Our lower bound is inspired by the work of Matoušek [31], which has also been utilized in the context of distance oracles [36]. Our arguments rely on the recent extremal combinatorics construction (see [12]) that was used to prove lower bounds on the size of distance approximating minors.

We start by reviewing a classical notion in combinatorial design.

**Definition 6.1** (Steiner Triple System). *Given a ground set  $T = [k]$ , an  $(3, 2)$ -Steiner system (abbr.  $(3, 2)$ -SS) of  $T$  is a collection of 3-subsets of  $T$ , denoted by  $\mathcal{S} = \{S_1, \dots, S_r\}$ , where  $r = \binom{k}{2} / 3$ , such that every 2-subset of  $T$  is contained in exactly one of the 3-subsets.*

**Lemma 6.2** ([38]). *For infinity many  $k$ , the set  $T = [k]$  admits an  $(3, 2)$ -SS.*

Roughly speaking, our proof proceeds by forming a  $k$ -terminal bipartite graph, where terminals lie on one side and non-terminals on the other. The set of non-terminals will correspond to some



subset of a Steiner Triple System  $\mathcal{S}$ , which will satisfy some *certain* property. One can equivalently view such a graph as taking union over *star* graphs. Before delving into details, we need to review a couple of other useful definitions and the construction from [12].

**Detour Graph and Cycle.** Let  $k$  be an integer such that  $T = [k]$  admits an  $(3, 2)$ -SS. Let  $\mathcal{S}$  be such an  $(3, 2)$ -SS. We associate  $\mathcal{S} = \{S_1, \dots, S_r\}$  with a graph whose vertex set is  $\mathcal{S}$ . We refer to such graph as a *detouring graph*. By the definition of Steiner system, it follows that  $|S_i \cap S_j|$  is either zero or one. Thus, two vertices  $S_i$  and  $S_j$  are adjacent in the detouring graph iff  $|S_i \cap S_j| = 1$ . It is also useful to label each edge  $(S_i, S_j)$  with the terminal in  $S_i \cap S_j$ . A *detouring cycle* is a cycle in the detouring graph such that no two neighbouring edges in the cycles have the same terminal label. Observe that the detouring graph has other cycles which are not detouring cycles.

Ideally, we would like to construct detouring graphs with long detouring cycles while keeping the size of the graph as large as possible. One trade-off is given in the following lemma.

**Lemma 6.3** ([12]). *For any integer  $t \geq 3$ , given a detouring graph with vertex set  $\mathcal{S}$ , there exists a subset  $\mathcal{S}' \subset \mathcal{S}$  of cardinality  $\Omega(k^{1+1/(t-1)})$  such that the induced graph on  $\mathcal{S}'$  has no detouring cycles of size  $t$  or less.*

Now we are ready to prove our incompressibility result regarding approximately preserving terminal pairwise distances.

**Proof of Theorem 1.2:** Let  $k$  be an integer such that  $T = [k]$  admits an  $(3, 2)$ -SS  $\mathcal{S}$ . Fix some integer  $t \geq 3$ , some positive constant  $c$  and use Lemma 6.3 to construct a subset  $\mathcal{S}'$  of  $\mathcal{S}$  of size  $\Omega(k^{1+1/(t-1)})$  such that the induced graph on  $\mathcal{S}'$  has no detouring cycles of size  $t$  or less. We may assume w.l.o.g. that  $\ell = |\mathcal{S}'| = c \cdot k^{1+1/(t-1)}$  (this can be achieved by repeatedly removing elements from  $\mathcal{S}'$ , as the property concerning the detouring cycles is not destroyed). Fix some ordering among 3-subsets of  $\mathcal{S}'$  and among terminals in each 3-subset.

We define the  $k$ -terminal graph  $G$  as follows:

- For each  $e_i \in \mathcal{S}'$  create a non-terminal vertex  $v_i$ . Let  $V_{\mathcal{S}'}$  denote the set of such vertices. The vertex set of  $G$  is  $T \cup V_{\mathcal{S}'}$ , where  $T = [k]$  denotes the set of terminals.
- For each  $e_i \in \mathcal{S}'$ , connect  $v_i$  to the three terminals  $\{x_1^i, x_2^i, x_3^i\}$  belonging to  $e_i$ , i.e., add edges  $(v_i, x_j^i)$ ,  $j = 1, 2, 3$ .

Note that  $G$  is sparse since both the number of vertices and edges are  $\Theta(\ell)$ , and it also holds that  $k = o(|V(G)|)$ .

For any subset  $R \subseteq \mathcal{S}'$ , we define the subgraph  $G_R = (V(G), E_R)$  of  $G$  as follows. For each  $e_i \in \mathcal{S}'$ , if  $e_i \in R$ , perform no changes. If  $e_i \notin R$ , delete the edge  $(v_i, x_1^i)$ . Note that there are  $2^\ell$  subgraphs  $G_R$ . We let  $\mathcal{G}$  denote the family of all such subgraphs.

We say a terminal pair  $(x, x')$  *respects*  $\mathcal{S}'$  if in the  $(3, 2)$ -SS  $\mathcal{S}$ , the unique 3-subset  $e$  that contains  $x$  and  $x'$  belongs to  $\mathcal{S}'$ . Given  $R \subseteq \mathcal{S}'$  and some terminal pair  $(x, x')$ , we say that  $R$  *covers*  $(x, x')$  if both  $x$  and  $x'$  are connected to some non-terminal  $v$  in  $G_R$ .

**Claim 6.4.** *For all  $R \subseteq \mathcal{S}'$  and terminal pairs  $(x, x')$  covered by  $R$  we have that  $d_{G_R}(x, x') = 2$ .*

*Proof.* By the definition of Steiner system and the construction of  $G_R$ , the shortest path between  $x$  and  $x'$  is simply a 2-hop path, i.e.,  $d_{G_R}(x, x') = 2$ .  $\square$

**Claim 6.5.** *For all  $R \subseteq \mathcal{S}'$  and any terminal pair  $(x, x')$  that respects  $\mathcal{S}'$  and is not covered by  $R$ , we have that  $d_{G_R}(x, x') \geq 2t$ .*

*Proof.* Since  $(x, x')$  respects  $\mathcal{S}'$ , there exists  $e_i = (x_1^i, x_2^i, x_3^i) \in \mathcal{S}'$  that contains both  $x$  and  $x'$ . By construction of  $G_R$  and the fact that  $(x, x')$  is not covered by  $R$ , it follows that  $e_i \in \mathcal{S}' \setminus R$ , and one of  $x, x'$  corresponds to  $x_1^i$  and the other corresponds to  $x_2^i$  or  $x_3^i$ . W.l.o.g., we assume  $x = x_1^i$  and  $x' = x_2^i$ . Note that there is no edge connecting  $x_1^i$  with the non-terminal  $v_i$  that corresponds to  $e_i$ . Now by Lemma 6.3, the detouring graph induced on  $\mathcal{S}'$  has no detouring cycles of size  $t$  or less, which implies that any other simple path between  $x_1^i$  and  $x_2^i$  in  $G$  must pass through at least  $t - 1$  other terminals. Let  $w_1, \dots, w_{t-1}$  be such terminals and let  $P := x_1^i \rightarrow w_1, \dots, w_{t-1} \rightarrow x_2^i$  denote the corresponding path, ignoring the non-terminals along the path. Between any consecutive terminal pairs in  $P$ , the shortest path is at least 2. Thus, the length of  $P$  is at least  $2t$ , i.e.,  $d_{G_R}(x_1^i, x_2^i) \geq 2t$ .  $\square$

Fix any two subsets  $R_1, R_2 \subseteq \mathcal{S}'$  with  $R_1 \neq R_2$ . It follows that there exists a 3-subset  $e_i = (x_1^i, x_2^i, x_3^i) \in \mathcal{S}'$  such that either  $e_i \in R_1 \setminus R_2$  or  $e_i \in R_2 \setminus R_1$ . Assume w.l.o.g. that  $e_i \in R_2 \setminus R_1$ . Note that  $(x_1^i, x_2^i)$  respects  $\mathcal{S}'$  and it is covered in  $R_2$  but not in  $R_1$ . By Claim 6.4 and 6.5, it holds that  $d_{G_{R_2}}(x_1^i, x_2^i) = 2$  and  $d_{G_{R_1}}(x_1^i, x_2^i) \geq 2t$ . In other words, there exists a set  $\mathcal{G}$  of  $2^\ell$  different subgraphs on the same set of nodes  $V(G)$  satisfying the following property: for any  $G_1, G_2 \in \mathcal{G}$ , there exists a terminal pair  $(x, x')$  such that the distances between  $x$  and  $x'$  in  $G_1$  and  $G_2$  differ by at least a  $t$  factor as well as by at least  $2t - 2$ . On the other hand, for any compression function that approximates terminal path distances within a factor of  $t - \varepsilon$  or an additive error  $2t - 3$  and produces a bitstring with less than  $\ell$  bits, there exist two different graphs  $G_1, G_2 \in \mathcal{G}$  that map to the same bit string. Hence, any such compression function must use at least  $\Omega(\ell) = \Omega(k^{1+1/(t-1)})$  bits if we want to preserve terminal distances within a  $t - \varepsilon$  factor or an additive error  $2t - 3$ .

To complete our argument, we need to show the claim for quality  $t = 2$ . The only significant modification we need is the usage of an  $(3, 2)$ -SS in the construction of graph  $G$  (instead of using a subset of it). The remaining details are similar to the above proof and we omit them here.

## References

- [1] Amir Abboud and Greg Bodwin. The  $4/3$  additive spanner exponent is tight. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*, pages 351–361. ACM, 2016.
- [2] Amir Abboud, Greg Bodwin, and Seth Pettie. A hierarchy of lower bounds for sublinear additive spanners. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 568–576. SIAM, 2017.
- [3] Alexandr Andoni, Anupam Gupta, and Robert Krauthgamer. Towards  $(1 + \varepsilon)$ -approximate flow sparsifiers. In *Proc. of the 25th SODA*, pages 279–293, 2014.
- [4] Baruch Awerbuch. Complexity of network synchronization. *Journal of the ACM (JACM)*, 32(4):804–823, 1985.
- [5] András A. Benczúr and David R. Karger. Approximating  $s$ - $t$  minimum cuts in  $\tilde{O}(n^2)$  time. In *Proc. of the 28th STOC*, pages 47–55, 1996.
- [6] T-H Hubert Chan, Donglin Xia, Goran Konjevod, and Andrea Richa. A tight lower bound for the steiner point removal problem on trees. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 70–81. Springer, 2006.

- [7] Moses Charikar, Tom Leighton, Shi Li, and Ankur Moitra. Vertex sparsifiers and abstract rounding algorithms. In *Proc. of the 51th FOCS*, pages 265–274, 2010.
- [8] Shiva Chaudhuri, KV Subrahmanyam, Frank Wagner, and Christos D Zaroliagis. Computing mimicking networks. *Algorithmica*, 26(1):31–49, 2000.
- [9] Chandra Chekuri, Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Embedding  $k$ -outerplanar graphs into  $l_1$ . *SIAM Journal on Discrete Mathematics*, 20(1):119–136, 2006.
- [10] Chandra Chekuri, Sanjeev Khanna, and F Bruce Shepherd. Edge-disjoint paths in planar graphs with constant congestion. *SIAM Journal on Computing*, 39(1):281–301, 2009.
- [11] Chandra Chekuri, F Bruce Shepherd, and Christophe Weibel. Flow-cut gaps for integer and fractional multiflows. In *Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms*, pages 1198–1208. Society for Industrial and Applied Mathematics, 2010.
- [12] Yun Kuen Cheung, Gramoz Goranci, and Monika Henzinger. Graph minors for preserving terminal distances approximately - Lower and Upper Bounds. In *Proc. of the 43rd ICALP*, pages 131:1–131:14, 2016.
- [13] Julia Chuzhoy. On vertex sparsifiers with steiner nodes. In *Proc. of the 44th STOC*, pages 673–688, 2012.
- [14] Edward B Curtis, David Ingerman, and James A Morrow. Circular planar graphs and resistor networks. *Linear algebra and its applications*, 283(1):115–150, 1998.
- [15] Matthias Englert, Anupam Gupta, Robert Krauthgamer, Harald Räcke, Inbal Talgam-Cohen, and Kunal Talwar. Vertex sparsifiers: New results from old techniques. *SIAM J. Comput.*, 43(4):1239–1262, 2014.
- [16] Thomas A Feo and J Scott Provan. Delta-wye transformations and the efficient reduction of two-terminal planar graphs. *Operations Research*, 41(3):572–582, 1993.
- [17] Isidoro Gitler. *Delta-Wye-Delta Transformations: Algorithms and Applications*. PhD thesis, Department of Combinatorics and Optimization, University of Waterloo, 1991.
- [18] Gramoz Goranci and Harald Räcke. Vertex sparsification in trees. In *Proc. of the 14th WAOA*, pages 103–115, 2016.
- [19] Anupam Gupta. Steiner points in tree metrics don’t (really) help. In *Proc. of the 12th SODA*, pages 220–227, 2001.
- [20] Torben Hagerup, Jyrki Katajainen, Naomi Nishimura, and Prabhakar Ragde. Characterizing multiterminal flow networks and computing flows in networks of small treewidth. *J. Comput. Syst. Sci.*, 57(3):366–375, 1998.
- [21] Lior Kamma, Robert Krauthgamer, and Huy L Nguyn. Cutting corners cheaply, or how to remove steiner points. *SIAM Journal on Computing*, 44(4):975–995, 2015.
- [22] Arindam Khan and Prasad Raghavendra. On mimicking networks representing minimum terminal cuts. *Inf. Process. Lett.*, 114(7):365–371, 2014.

- [23] Robert Krauthgamer, Huy L Nguyen, and Tamar Zondiner. Preserving terminal distances using minors. *SIAM J. Discrete Math.*, 28(1):127–141, 2014.
- [24] Robert Krauthgamer and Inbal Rika. Mimicking networks and succinct representations of terminal cuts. In *Proc. of the 24th SODA*, pages 1789–1799, 2013.
- [25] Robert Krauthgamer and Tamar Zondiner. Preserving terminal distances using minors. In *Proc. of the 39th ICALP*, pages 594–605, 2012.
- [26] Felix Lazebnik, Vasiliy A Ustimenko, and Andrew J Woldar. A new series of dense graphs of high girth. *Bulletin of the American mathematical society*, 32(1):73–79, 1995.
- [27] Felix Lazebnik, Vasiliy A Ustimenko, and Andrew J Woldar. A characterization of the components of the graphs  $d(k, q)$ . *Discrete Mathematics*, 157(1-3):271–283, 1996.
- [28] James R Lee, Manor Mendel, and Mohammad Moharrami. A node-capacitated okamura-seymour theorem. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 495–504. ACM, 2013.
- [29] Frank Thomson Leighton and Ankur Moitra. Extensions and limits to vertex sparsification. In *Proc. of the 42nd STOC*, pages 47–56, 2010.
- [30] Konstantin Makarychev and Yury Makarychev. Metric extension operators, vertex sparsifiers and lipschitz extendability. In *Proc. of the 51th FOCS*, pages 255–264, 2010.
- [31] Jiří Matoušek. On the distortion required for embedding finite metric spaces into normed spaces. *Israel Journal of Mathematics*, 93(1):333–344, 1996.
- [32] Ankur Moitra. Approximation algorithms for multicommodity-type problems with guarantees independent of the graph size. In *Proc. of the 50th FOCS*, 2009.
- [33] Haruko Okamura and Paul D. Seymour. Multicommodity flows in planar graphs. *Journal of Combinatorial Theory, Series B*, 31(1):75 – 81, 1981.
- [34] Daniel A. Spielman and Shang-Hua Teng. Spectral sparsification of graphs. *SIAM J. Comput.*, 40(4):981–1025, 2011.
- [35] Roberto Tamassia and Ioannis G Tollis. Planar grid embedding in linear time. *IEEE Trans. Circuits Syst.*, 36(9):1230–1234, 1989.
- [36] Mikkel Thorup and Uri Zwick. Approximate distance oracles. *Journal of the ACM (JACM)*, 52(1):1–24, 2005.
- [37] Leslie G. Valiant. Universality considerations in VLSI circuits. *IEEE Trans. Computers*, 30(2):135–140, 1981.
- [38] Richard M. Wilson. An existence theory for pairwise balanced designs, III: proof of the existence conjectures. *J. Comb. Theory, Ser. A*, 18(1):71–79, 1975.
- [39] David P Woodruff. Lower bounds for additive spanners, emulators, and more. In *Foundations of Computer Science, 2006. FOCS'06. 47th Annual IEEE Symposium on*, pages 389–398. IEEE, 2006.